Large time behaviour for a viscous Hamilton-Jacobi equation with Neumann boundary condition

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Abstract

We prove the existence and the uniqueness of strong solutions for the viscous Hamilton-Jacobi equation: $u_t - \Delta u = a|\nabla u|^p,\ t > 0,\ x \in \Omega$ with Neumann boundary condition, and initial data μ_0, a continuous function. The domain Ω is a bounded and convex open set with smooth boundary, $a \in \mathbb{R}, a \neq 0$ and $p > 0$. Then, we study the large time behavior of the solution and we show that for $p \in (0,1)$, the extinction in finite time of the gradient of the solution occurs, while for $p \geq 1$ the solution converges uniformly to a constant, as $t \to \infty$.

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1 Introduction and main results

Consider the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = a|\nabla u|^p & \text{in} \quad (0,+\infty)\times\Omega, \\ \dfrac{\partial u}{\partial \nu}(t,x) = 0 & \text{on} \quad (0,+\infty)\times\partial\Omega, \\ u(0,x) = \mu_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $a \in \mathbb{R}$, $a \neq 0$, $p > 0$ and $\Omega \subset \mathbb{R}^N$ is a bounded open set with smooth boundary of C^3 class.

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The Cauchy problem in the whole space $\mathbb{R}^N$ has been intensively studied (see [2, 3, 5, 9, 10, 17, 25]). As well, in bounded domains $\Omega \subset \mathbb{R}^N$, existence and uniqueness results of the solutions for the Cauchy-Dirichlet problem have been obtained in [1, 6, 12, 24]. In particular the large time behavior of the solution to the Cauchy problem has been analysed in [4, 7, 8], as $a < 0$ and for initial data μ_0 a positive function. Thus, in [8], we can find the following result: if $a < 0$, $p \in (0, 1)$ and the initial data μ_0 is a periodic function, the extinction in finite time of the solution of problem (1.1) occurs. Since, any positive solution of the Cauchy problem is a super-solution of the homogeneous Cauchy-Dirichlet problem, the result of [8], remain valid also in bounded domains for the Cauchy-Dirichlet problem.

With respect to the Cauchy-Neumann we mention the results given in [13], regarding the existence, uniqueness and regularity of weak solutions, for $p \in (0, 2)$, $a \in \mathbb{R}, a \neq 0$ and initial data μ_0 a bounded Radon measure or a measurable function in $L^q(\Omega)$, $q \geq 1$. To our knowledge the problem (1.1) has not been investigated for the super-quadratic case, $p \geq 2$.

In this paper we consider the problem (1.1) when Ω is a bounded and convex open set, and we give some existence and uniqueness results of the solutions when the initial data is a continuous function in $\overline{\Omega}$. Then we study the large time behavior of the solutions according to the exponent p. The results rely on some remarkable estimates for the gradient of the solutions of problem (1.1), obtained by using a Bernstein technique. These estimates, given in Theorem 1.2 are used as the key argument in the proof of the extinction result in Theorem 1.3. More exactly we show that: if $p \in (0, 1)$ then, for any solution u of problem (1.1) with initial data in $C(\overline{\Omega})$ there exists $T^* > 0$ and $c \in \mathbb{R}$ such that:

$$u(t, x) \equiv c, \text{ for all } t > T^* \text{ and } x \in \Omega.$$

This property is called: "the extinction of the gradient of the solution u in finite time". Also, in Theorem 1.3 we prove that, for $p \geq 1$ any solution of problem (1.1) converges uniformly to a constant, as $t \to \infty$.

The notations used are mostly standard for the parabolic equations theory:
For all $0 < \tau < T \leq \infty$ we denote by $Q_T = (0, T) \times \Omega$, $\Gamma_T = (0, T) \times \partial\Omega$, $Q_{\tau,T} = (\tau, T) \times \Omega$ and $\Gamma_{\tau,T} = (\tau, T) \times \partial\Omega$. $C(\overline{\Omega})$ is the space of continuous functions on $\overline{\Omega}$. $C_b(\Omega)$ is the space of bounded continuous functions on Ω. $C_0(\overline{\Omega})$ the space of continuous functions on $\overline{\Omega}$ which vanish on the boundary $\partial\Omega$. $C_c^\infty(\Omega)$ (resp. $C_c^\infty(Q_T)$) the space of infinitely differentiable functions on Ω (resp. Q_T) with compact support in Ω (resp. Q_T). $C^{0,1}([0, T) \times \overline{\Omega})$ is the space of continuous functions u on $[0, T) \times \overline{\Omega}$ which are differentiable with respect to $x \in \Omega$ and the derivatives $(\frac{\partial u}{\partial x_i})_{1 \leq i \leq N}$ are in $C([0, T) \times \overline{\Omega})$. $C^{1,2}(Q_T)$ is the space of continuous functions u on Q_T such that the derivatives: $\frac{\partial u}{\partial t}$, $(\frac{\partial u}{\partial x_i})_{1 \leq i \leq N}$, $(\frac{\partial u}{\partial x_i \partial x_j})_{1 \leq i,j \leq N}$, exist and belong to $C(Q_T)$. Suppose that α is a positive real number and $[\alpha]$ the integer

part of α such that $[\alpha] < \alpha$ then: $C^{\alpha}(\overline{\Omega})$ and $C^{\alpha/2,\alpha}(\overline{Q})$ denote the usual Hölder spaces on the bounded open sets $\Omega \subset \mathbb{R}^N$ and $Q \subset \mathbb{R}^{N+1}$ respectively (for the definitions see [16, 20]).
We denote by $\mathcal{M}_b(\Omega)$ the space of bounded Radon measures on Ω endowed with the usual norm $\| \|_{\mathcal{M}_b(\Omega)}$. For $q \geq 1$, $\| \|_q$ is the usual norm of the Lebesgue space $L^q(\Omega)$. $W^{1,q}(\Omega)$, $W^{1,q}(Q_T)$ and $W_q^{1,2}(Q_T)$ are the usual Sobolev spaces in Ω respectively Q_T (for the definitions see [21]).
We denote by $(S(t))_{t\geq 0}$ the semigroup of contraction in $L^q(\Omega), q \geq 1$ related to the heat equation with homogeneous Neumann boundary condition (see [23]). As we can see in [13] this semigroup can be extended, in a natural way, to the space of bounded Radon measures, $\mathcal{M}_b(\Omega)$.

First we recall an existence and uniqueness result for the solutions of problem (1.1) when $p \in (0, 2)$ (for further details see [13]).

Theorem 1.1 *[13] Let $\mu_0 \in \mathcal{M}_b(\Omega)$. Then (1.1) admits a weak solution $u \in L^{\infty}(0, T; L^1(\Omega)) \cap L^1(0, T; W^{1,1}(\Omega)) \cap C^{1+\delta/2,2+\delta}(\overline{Q_{\tau,T}})$, $T > 0, \tau \in (0, T)$, $\delta \in (0, 1)$, such that $|\nabla u|^p \in L^1(Q_T)$, in the following cases:*

i) $0 < p < 2/(N + 1)$. *The solution is unique if Ω is convex.*

ii) $2/(N + 1) \leq p < 1$. *The solution is unique if $\mu_0 \in L^q(\Omega)$ for some $q > pN/(2 - p)$ and Ω a convex open set.*

iii) $1 \leq p < (N + 2)/(N + 1)$. *The solution is unique. If $\mu_0 \in L^q(\Omega)$ for some $q \geq 1$ then $u \in C([0, T]; L^q(\Omega)) \cap L^p(0, T; W^{1,pq}(\Omega))$.*

iv) $(N + 2)/(N + 1) \leq p < 2$, *$\mu_0 \in L^q(\Omega)$ and $q > q_c = \frac{N(p-1)}{2-p}$. There holds $u \in C([0, T]; L^q(\Omega)) \cap L^p(0, T; W^{1,pq}(\Omega))$ and the solution is unique in this space.*

v) $(N + 2)/(N + 1) \leq p < 2$, $\mu_0 \in L^1(\Omega)$, $\mu_0 \geq 0$.

Moreover, this solution satisfies (1.1) in the mild sense:

$$u(t) = S(t)\mu_0 + a \int_0^t S(t - s)|\nabla u(s)|^p \, ds, \quad t \in (0, T).$$

In Theorem 1.2 below we prove the existence and the uniqueness of solutions of problem (1.1), for $p > 0$, Ω a bounded and convex open set with smooth boundary, and for initial data $\mu_0 \in C(\overline{\Omega})$. We give also some gradient estimates of the solution u of problem (1.1) which will be very useful in the proof of Theorem 1.3.

Let u be a function in $C(\overline{Q_\infty})$. For any $t \geq 0$ denote by:

$$M(t) = \max_{x\in\overline{\Omega}} u(t, x) \tag{1.2}$$

and

$$m(t) = \min_{x \in \overline{\Omega}} u(t,x), \tag{1.3}$$

Theorem 1.2 *Consider $a \in \mathbb{R}, a \neq 0$, $p > 0$ and $\mu_0 \in C(\overline{\Omega})$, where Ω is a bounded and convex open set. Then, the problem (1.1) admits a unique solution:*

$$u \in C(\overline{Q_T}) \cap C^{1+\delta/2, 2+\delta}(\overline{Q_{\tau,T}})$$

for any $T > 0$ and $\tau \in (0,T)$. Moreover, we have:

$$t \to M(t) \text{ is a decreasing function in } \mathbb{R}, \tag{1.4}$$

$$t \to m(t) \quad \text{is a non-decreasing function in } \mathbb{R}, \tag{1.5}$$

$$\|\nabla u(t)\|_\infty \leq \left(\frac{1}{2}\right)^{1/2} (M(s) - m(s))(t-s)^{-\frac{1}{2}} \text{ for all } t > s \geq 0, \tag{1.6}$$

and for $p \neq 1$

$$\|\nabla u(t)\|_\infty \leq \left(\frac{\max\{p,2\}}{ap|1-p|}\right)^{1/p} (M(s) - m(s))^{1/p}(t-s)^{-1/p} \text{ for all } t > s \geq 0. \tag{1.7}$$

For the proof we are using the Bernstein technique. This method can be found in [9, 12, 17] and [22], where formulas similar to (1.6) and (1.7) are obtained for the Cauchy problem in $\mathbb{R}^N$. This method has also been used by Ph. Benilan [11] in order to obtain remarkable estimates for the solutions of "the porous medium equation"

In the next result we are going to analyze the large time behavior of the solutions for problem (1.1).

Theorem 1.3 *Consider $a \in \mathbb{R}, a \neq 0$, $p > 0$ and Ω a bounded and convex domain. Let $\mu_0 \in C(\overline{\Omega})$ and denote by u a solution of problem (1.1) corresponding to μ_0. Then:*

i) If $p \in (0,1)$, the extinction of the gradient of u in finite time occurs, in other words:
there exists $T^* \in [0, +\infty)$ and $c \in \mathbb{R}$ such that:

$$u(t,x) \equiv c \text{ for all } t \geq T^* \text{ and } x \in \overline{\Omega},$$

ii) If $p \in [1, +\infty)$, then $u(t, \cdot)$ converges uniformly on $\overline{\Omega}$ to a constant, as $t \to \infty$. Moreover the decreasing rate is given by:

$$M(t) - m(t) \leq \left(\frac{8f(t/2)}{t^2} \right)^{\frac{1}{\gamma}}, \quad \forall\ t > 0,$$

where f is defined in (4.22) and γ in (4.2).

Remark 1.4 *From (4.22) we have:*

$$\left(\frac{8f(t/2)}{t^2} \right)^{\frac{1}{\gamma}} = \begin{cases} C_1 t^{-\frac{2}{\gamma}} e^{-C_2 t} & \text{if} \quad p = 1, \\ C_3 t^{-\frac{\alpha+1}{\gamma(\alpha-1)}} & \text{if} \quad p > 1. \end{cases}$$

where γ and α are given by (4.2) and (4.5), and C_1, C_2, C_3 are positive constants which depends only on p, N, γ.

The proof of Theorem 1.3 follows the same ideas as in [8]. In this paper, the authors investigate the large time behaviour for the Cauchy problem in the whole space $\mathbb{R}^N$ and for initial data periodic functions. We mention that the key arguments of the proof are the relations (1.6) and (1.7) above.

Remark 1.5 *Theorem 1.3 is valid for any $a \in \mathbb{R}, a \neq 0$, while in [7] and [8] the result is proved for $a < 0$.*

The next result is a simple consequence of Theorem 1.1 and Theorem 1.3 above.

Corollary 1.6 *Let Ω be a bounded and convex domain with smooth boundary and let $a \in \mathbb{R}, a \neq 0$. Then:*

i) If $p \in (0, 1)$ and μ_0 is a bounded Radon measure, the extinction in finite time of the gradient of any weak solution u of problem (1.1) occurs.

ii) The weak solution $u(t, \cdot)$ of problem (1.1) converges uniformly in $\overline{\Omega}$, to a constant $c \in \mathbb{R}$, as $t \to \infty$, in the two cases below:
a) $p \in [1, \frac{N+2}{N+1})$ and μ_0 is a bounded Radon measure,
b) $p \in [\frac{N+2}{N+1}, 2)$ and $\mu_0 \in L^q(\Omega)$, $q > q_c = \frac{N(p-1)}{2-p}$.

This paper is organized as follows: In section 2, we give some preliminary results. In section 3, we introduce the technique of Bernstein to obtain some uniform estimates for the gradient of the solution of problem (1.1) and we prove the Theorem 1.2.Finally section 4 is devoted to the proof of Theorem 1.3, which concerns the large time behaviour of solutions.

2 Preliminary results

We start with some auxiliary results.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and consider $\mu_0 \in C(\overline{\Omega})$. Denote by $m = \min_{x\in\Omega} \mu_0(x)$ and $M = \max_{x\in\Omega} \mu_0(x)$.*
Then, there exists a sequence $(u_0^n)_{n\geq 1} \subset C^{3+\beta}(\overline{\Omega}), (\beta \in (0,1))$ such that:

$$u_0^n \searrow \mu_0 \text{ as } n \to \infty, \tag{2.1}$$

$$m + \frac{1}{2^{n+1}} \leq u_0^n \leq M + \frac{1}{2^{n-1}}, \ \forall n \geq 1, \tag{2.2}$$

and

$$\frac{\partial u_0^n}{\partial \nu} = 0 \text{ on } \partial\Omega. \tag{2.3}$$

Proof:
For any $n \in \mathbb{N}^*$, denote by $v_0^n = \mu_0 + \frac{1}{2^n}$, then $(v_0^n)_n \subset C(\overline{\Omega})$. For $t > 0$ let us set:

$$v^n(t) = S(t)v_0^n.$$

Then $v^n \in C(\overline{Q_\infty}) \cap C^\infty(\overline{Q_{\tau,\infty}})$ for all $\tau \in (0,\infty)$, and:

$$\frac{\partial v^n}{\partial \nu}(t,x) = 0 \quad \text{for all } (t,x) \in \Gamma_T.$$

Since $v^n \in C(\overline{Q_\infty})$, there exists t_n close enough from 0 such that:

$$|v^n(t_n,x) - v_0^n(x)| < \frac{1}{2^{n+2}}, \qquad \forall x \in \Omega. \tag{2.4}$$

Denote by:

$$u_0^n(x) = v^n(t_n,x), \qquad x \in \Omega.$$

Then $u_0^n \in C^\infty(\overline{\Omega})$ and satisfies condition (2.3). Moreover, thanks to (2.4) we have on the one hand:

$$u_0^n - \mu_0 = (u_0^n - v_0^n) + (v_0^n - \mu_0) \leq \frac{1}{2^{n+2}} + \frac{1}{2^n} \leq \frac{1}{2^{n-1}},$$

on the other hand:

$$u_0^n - \mu_0 = (u_0^n - v_0^n) + (v_0^n - \mu_0) \geq -\frac{1}{2^{n+2}} + \frac{1}{2^n} \geq \frac{1}{2^{n+1}}.$$

which yields (2.2).
To prove that $(u_0^n)_n$ is a decreasing sequence, let compute:

$$u_0^n - u_0^{n+1} = (u_0^n - v_0^n) + (v_0^n - v_0^{n+1}) + (v_0^{n+1} - u_0^{n+1}) \geq -\frac{1}{2^{n+2}} + \frac{1}{2^{n+1}} - \frac{1}{2^{n+3}} = \frac{1}{2^{n+3}} > 0.$$

And finally we obtain (2.1). $\square$

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^N$ be a convex and bounded domain and q a real number such that $q > N$. From the Sobolev embedding, $W^{1,q}(\Omega) \hookrightarrow C(\overline{\Omega})$, for all $u \in W^{1,q}(\Omega)$, the following quantities:*

$$M_u = \max_{x \in \overline{\Omega}} u(x) \quad \textit{and } m_u = \min_{x \in \overline{\Omega}} u(x),$$

are well defined. Moreover we have :

$$M_u - m_u \leq C \|\nabla u\|_q, \tag{2.5}$$

where C is a positive constant depending only on q, N and Ω.

Proof: The proof is similar to that of Lemmas 7.16 and 7.17 in [14]. Ω being a convex set, for all $x, y \in \Omega$ we have $(1-t)x + ty \in \Omega$ for any $t \in [0,1]$. Let $u \in W^{1,q}(\Omega)$, then:

$$u(x) - u(y) = \int_0^1 \nabla u((1-t)x + ty) \cdot (x - y)\, dt,$$

which yields:

$$u(x) - \frac{1}{|\Omega|} \int_\Omega u(y)\, dy = \frac{1}{|\Omega|} \int_\Omega \int_0^1 \nabla u((1-t)x + ty) \cdot (x - y)\, dt\, dy.$$

Denote by:

$$u_\Omega = \frac{1}{|\Omega|} \int_\Omega u(y)\, dy \qquad \text{and} \qquad d = diam(\Omega).$$

Then,

$$|u(x) - u_\Omega| \leq \frac{d}{|\Omega|} \int_\Omega \int_0^1 |\nabla u((1-t)x + ty)| dt dy \leq \frac{d}{|\Omega|} \int_0^1 \int_\Omega |\nabla u((1-t)x + ty)| dy dt.$$

We replace $(1-t)x + ty = \zeta$ and, for any $t \in [0,1]$, we denote by Ω_t the set:

$$\Omega_t = \{\zeta = (1-t)x + ty ; y \in \Omega\} \subset \Omega,$$

then:

$$|u(x) - u_\Omega| \leq \frac{d}{|\Omega|} \int_0^1 \int_{\Omega_t} |\nabla u(\zeta)| t^{-N} dt\, d\zeta.$$

Using the Hölder inequality for $q > N$ we get:

$$
\begin{aligned}
|u(x) - u_\Omega| &\le \frac{d}{|\Omega|} \int_0^1 \Big[\int_\Omega |\nabla u(\zeta)|^q \, d\zeta \Big]^{1/q} |\Omega_t|^{1-1/q} t^{-N} \, dt \\
&\le \frac{d}{|\Omega|} \|\nabla u\|_q \cdot \int_0^1 t^{N(1-1/q)} t^{-N} |\Omega|^{1-1/q} \, dt \\
&\le \frac{d}{|\Omega|^{1/q}} \|\nabla u\|_q \cdot \int_0^1 t^{-N/q} \, dt \le \frac{d}{|\Omega|^{1/q}} \frac{q}{q-N} \|\nabla u\|_q.
\end{aligned}
$$

Finally, for $x, y \in \Omega$ we obtain:

$$
|u(x) - u(y)| \le |u(x) - u_\Omega| + |u_\Omega - u(y)| \le \frac{2d}{|\Omega|^{1/q}} \cdot \frac{q}{q-N} \|\nabla u\|_q,
$$

and relation (2.5) follows. Thus, the Lemma 2.2 is achieved. $\square$

Lemma 2.3 *Let $\Omega \subset \mathbb{R}^N$ be a convex and bounded domain, then for $u \in C^2(\overline{\Omega})$ such that $\dfrac{\partial u}{\partial \nu} |_{\partial\Omega} = 0$, we have:*

$$
\frac{\partial}{\partial \nu} |\nabla u|^2 \le 0 \ \text{on} \ \partial\Omega.
$$

For the proof see Lemma I.1, p. 350 in [22].

The following lemma is a comparison principle for parabolic nonlinear equations, which generalize the result obtained in [19], to less regular functions.

Lemma 2.4 *Let $\Omega \subset \mathbb{R}^N$ be a convex and bounded open set with smooth boundary and denote by $\mathcal{N}$, the nonlinear parabolic operator, defined by:*

$$
\mathcal{N}(u) = \frac{\partial u}{\partial t} - \Delta u - f(t, x, u, \nabla u)
$$

where f is a uniformly continuous function satisfying:
for all $r > 0$ there exists $L_r > 0$ such that:

$$
\begin{aligned}
&|f(t, x, y_1, v_1) - f(t, x, y_2, v_2)| \le L_r(|y_1 - y_2| + |v_1 - v_2|), \\
&\qquad \text{for all } (t, x) \in Q_T \text{ and } \ y_1, y_2 \in (-r, r), \ v_1, v_2 \in B_r(0),
\end{aligned}
\tag{2.6}
$$

where:

$$
B_r(0) = \{\xi \in \mathbb{R}^N; \ |\xi| < r\}.
$$

Let u^1 and u^2 be two functions in $C^{0,1}(\overline{Q_T}) \cap C^{1,2}(Q_T)$, such that:

$$\begin{cases} \mathcal{N}(u^1)(t,x) \leq 0 \leq \mathcal{N}(u^2)(t,x) \text{ for all } (t,x) \in Q_T \\ \dfrac{\partial u^1}{\partial \nu} \leq \dfrac{\partial u^2}{\partial \nu} \text{ on } \Gamma_T \\ u^1(0,x) \leq u^2(0,x) \text{ for all } x \in \Omega \end{cases} \tag{2.7}$$

Then

$$u^1 \leq u^2 \text{ on } Q_T.$$

We begin the proof by the following useful remark:

Remark 2.5 *Let Ω be a convex open set in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, which contains the origin. For $x \in \partial\Omega$, denote by $\nu(x)$ the unit outward normal on $\partial\Omega$ at the point x. Then:*

$$x \cdot \nu(x) > 0.$$

Proof of Lemma 2.4: Supposing first that Ω is a convex open set which contain the origin and denoting by

$$R = \max\{\sup_{(t,x)\in Q_T} |u_1(t,x)|; \sup_{(t,x)\in Q_T} |\nabla u_1|(t,x); \sup_{(t,x)\in Q_T} |u_2(t,x)|; \sup_{(t,x)\in Q_T} |\nabla u_2|(t,x)\},$$

then, from (2.6), there exists $L_R > 0$ such that:

$$|f(t,x,u_1,\nabla u_1) - f(t,x,u_2,\nabla u_2)| \leq L_R(|u_1-u_2| + |\nabla u_1 - \nabla u_2|), \quad (t,x) \in Q_T. \tag{2.8}$$

For any $\varepsilon \in (0,1)$ consider the function:

$$z(t,x) = u_1(t,x) - u_2(t,x) - \varepsilon e^{Ct}(1+|x|^2)^{\frac{1}{2}},$$

where $C = 2L_R + N$. Then, using the regularity of u_1 and u_2 we deduce that:

$$z \in C^{0,1}(\overline{Q_T}) \cap C^{1,2}(Q_T).$$

For any $t \in [0,T]$ let us define the function:

$$\varphi(t) = \max\{\sup_{x\in\Omega} z(t,x); 0\},$$

then $\varphi \in C([0,T])$, and for any $t \in (0,T]$ we can define:

$$\overline{\varphi'}(t) = \limsup_{h\searrow 0} \frac{\varphi(t) - \varphi(t-h)}{h}.$$

Thus, in order to prove that:

$$z \leq 0 \text{ in } Q_T, \tag{2.9}$$

we need to show that:

$$\overline{\varphi'}(t) \leq L_R \varphi(t) \text{ for all } \ t \in (0,T). \tag{2.10}$$

Indeed, as $\varphi(0) = 0$ and $\varphi \geq 0$, we can apply Theorem 4.1 in [18] to the differential inequality (2.10) and we deduce that $\varphi \equiv 0$. Which implies (2.9).
Proof of (2.10): Consider $t \in (0,T]$.
There are two possibilities. Either $\varphi(t) = 0$ and (2.10) holds because, in this case, $\overline{\varphi'}(t) \leq 0$. Or $\varphi(t) > 0$, and in particular, there exists $x_0 \in \overline{\Omega}$ such that:

$$z(t,x_0) = \varphi(t) > 0.$$

We claim that $x_0 \notin \partial\Omega$. Indeed, if $x_0 \in \partial\Omega$, on the one hand:

$$\frac{\partial z}{\partial \nu}(t,x_0) = \lim_{\lambda \nearrow 0} \frac{z(t,x_0+\lambda\nu) - z(t,x_0)}{\lambda} \geq 0.$$

On the other hand, thanks to hypothesis (2.7) and to Remark 2.5 we have:

$$\frac{\partial z}{\partial \nu} = \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} - \varepsilon e^{Ct} \frac{x \cdot \nu}{(1+|x|^2)^{\frac{1}{2}}} \leq -\varepsilon e^{Ct} \frac{x \cdot \nu}{(1+|x|^2)^{\frac{1}{2}}} < 0 \text{ on } \Gamma_T.$$

So, we have a contradiction. Consequently, $x_0 \in \Omega$ is a positive maximum point for the function $\Omega \ni x \mapsto z(t,x)$. In particular we have:

$$\nabla z(t,x_0) = 0 \ \text{ and } \ \Delta z(t,x_0) \leq 0. \tag{2.11}$$

Since, for any $h > 0$, $z(t-h,x_0) \leq \varphi(t-h)$, we deduce:

$$\overline{\varphi'}(t) \leq \lim_{h \nearrow 0} \frac{z(t,x_0) - z(t-h,x_0)}{h} = \frac{\partial z}{\partial t}(t,x_0). \tag{2.12}$$

On the other hand, thanks to (2.8) and (2.11), at (t,x_0), we have:

$$\begin{aligned}
&\frac{\partial z}{\partial t}(t,x_0) = \frac{\partial u_1}{\partial t}(t,x_0) - \frac{\partial u_2}{\partial t}(t,x_0) - \varepsilon C e^{Ct}(1+|x_0|^2)^{\frac{1}{2}} \\
&\leq \Delta(u_1-u_2)(t,x_0) + f(t,x_0,u_1,\nabla u_1) - f(t,x_0,u_2,\nabla u_2) - \varepsilon C e^{Ct}(1+|x_0|^2)^{\frac{1}{2}} \\
&\leq \Delta z + \varepsilon e^{Ct} \tfrac{(N+(N-1)|x_0|^2)}{(1+|x_0|^2)^{3/2}} + L_R|u_1-u_2| + L_R|\nabla u_1 - \nabla u_2| - \varepsilon C e^{Ct}(1+|x_0|^2)^{\frac{1}{2}} \\
&\leq \varepsilon e^{Ct}\Big(N + L_R \frac{|x_0|}{(1+|x_0|^2)^{\frac{1}{2}}}\Big) + L_R z(t,x_0) - \varepsilon(L_R+N)e^{Ct}(1+|x_0|^2)^{\frac{1}{2}} \\
&\leq L_R z(t,x_0) + \varepsilon(N+L_R)e^{Ct} - \varepsilon(N+L_R)e^{Ct}(1+|x_0|^2)^{\frac{1}{2}} \leq L_R\varphi(t).
\end{aligned} \tag{2.13}$$

Recall that $C = 2L_R + N$ and $z(t,x_0) = \varphi(t)$. Combining (2.12) and (2.13) we deduce (2.10). Thus (2.9) holds. We may let $\varepsilon \searrow 0$ in (2.9) and we get:

$$u_1 \le u_2 \text{ in } \ Q_T.$$

For the general case when Ω do not contains the origin, it is possible to translate the problem on a domain which contains the origin since the first equation of (1.1) is invariant to the translation. For example we can carry the study of the problem on $\Omega_{x_0} = \Omega - x_0$, where $x_0 \in \Omega$. $\square$

In the sequel we denote by $G : (0,+\infty)\times\Omega\times\Omega$ the heat kernel for the homogeneous Neumann boundary value problem, then, for fix $y \in \Omega$, $G(\cdot,\cdot,y)$ verifies:

$$\begin{cases} \dfrac{\partial G}{\partial t}(t,x,y) = \Delta_x G(t,x,y) \ \text{ in } \ Q_\infty, \\ \dfrac{\partial G}{\partial \nu}(t,x,y) = 0 \quad \text{ on } \ \Gamma_\infty, \\ G(t,x,y) \underset{t\to 0}{\rightharpoonup} \delta_y(x) \ \text{ weakly in } \mathcal{M}_b(\Omega). \end{cases}$$

The proof of the following property on the heat kernel can be found in [15, 16].

Lemma 2.6 *[15, 16] Let Ω be a bounded open set with smooth boundary and G the heat kernel for the homogeneous Neumann boundary value problem. Then for any $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^N$, and for any $T > 0$, there exists two positive constants $c > 0$ and $C(T) > 0$ such that:*

$$|D_x^\alpha D_t^l G(t,x,y)| \le C(T) t^{-(\frac{N}{2}+\frac{|\alpha|}{2}+l)} e^{-c\frac{|x-y|^2}{t}} \tag{2.14}$$

for all $(t,x,y) \in (0,T)\times\Omega\times\Omega$.

Consider, $\mu_0 \in L^\infty(\Omega)$ and $S(t)\mu_0$ the solution of the heat equation with initial data μ_0 and with homogeneous Neumann boundary condition. Then:

$$S(t)\mu_0(x) = \int_\Omega G(t,x,y)\mu_0(y)\,dy.$$

Thanks to (2.14), for any $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^N$ and for any $T > 0$ we have:

$$\|D_x^\alpha D_t^l S(t)\mu_0\|_\infty \le C(T)\|\mu_0\|_\infty t^{-(\frac{|\alpha|}{2}+l)}, \tag{2.15}$$

where $C(T)$ is a positive constant.

3 Proof of Theorem 1.2

We prove the theorem for $a > 0$. If $a < 0$, then $-a > 0$ and we notice that if v is the solution of problem (1.1) with initial data $-\mu_0$ instead of μ_0 and $-a$ instead of a, then $u = -v$ is the solution of problem (1.1) corresponding to data a and μ_0.

The proof follows five steps:

First step: "Smoothing".

Consider $\mu_0 \in C(\overline{\Omega})$ and denote by $M(0) = \max_{x\in\overline{\Omega}} \mu_0(x)$ and $m(0) = \min_{x\in\overline{\Omega}} \mu_0(x)$. Then, from Lemma 2.1, there exists a sequence of functions $(u_0^n)_{n\geq 1}$ satisfying:

$$\begin{cases} u_0^n \searrow \mu_0 \quad \text{as} \quad n \to \infty, \\ \dfrac{\partial u_0^n}{\partial \nu} = 0 \\ m(0) + \frac{1}{n} \leq u_0^n \leq M(0) + \frac{2}{n}, \quad \forall n \geq 1. \end{cases} \tag{3.1}$$

As in [9] and [17] we need to introduce a smooth function, related to $\xi \to a|\xi|^p$. So, for any $\varepsilon \in (0,1)$ we consider the application $F_\varepsilon : \mathbb{R}^N \to \mathbb{R}$ defined by:

$$F_\varepsilon(\xi) = \begin{cases} a(\varepsilon + |\xi|^2)^{p/2} & \text{if} \quad 0 < p \leq 1, \\ a(-\varepsilon + |\xi|^2)(\varepsilon + |\xi|^2)^{\frac{p-2}{2}} & \text{if} \quad 1 < p < 2, \\ a|\xi|^p & \text{if} \quad p \geq 2. \end{cases} \tag{3.2}$$

With $\rho > 0$ fixed, let us show that for any $\xi_1, \xi_2 \in B_\rho(0)$ and $\varepsilon \in (0,1)$ we have:

$$|F_\varepsilon(\xi_1) - F_\varepsilon(\xi_2)| \leq K\rho^{\max\{p-1,0\}}|\xi_1 - \xi_2|^{\min\{p,1\}}, \tag{3.3}$$

where K is a positive constant depending only on p and a.

To prove (3.3) we can distinguish among the three cases. So, using the Mean Value Theorem there exists $\lambda \in [0,1]$ such that:

The case $0 < p \leq 1$:

$$\begin{aligned} |F_\varepsilon(\xi_1) - F_\varepsilon(\xi_2)| &= a[(\varepsilon + |\xi_1|^2)^{p/2} - (\varepsilon + |\xi_2|^2)^{p/2}] \leq a[(\varepsilon + |\xi_1|^2)^{1/2} - (\varepsilon + |\xi_2|^2)^{1/2}]^p \\ &\leq a\left(\frac{|\lambda\xi_1 + (1-\lambda)\xi_2|}{(\varepsilon + |\lambda\xi_1 + (1-\lambda)\xi_2|^2)^{1/2}}|\xi_1 - \xi_2|\right)^p \leq a|\xi_1 - \xi_2|^p. \end{aligned}$$

The case $1 < p < 2$:

$$\begin{aligned} |F_\varepsilon(\xi_1) - F_\varepsilon(\xi_2)| &\leq |\nabla F_\varepsilon(\lambda\xi_1 + (1-\lambda)\xi_2) \cdot (\xi_1 - \xi_2)| \\ &\leq a\frac{2|\lambda\xi_1 + (1-\lambda)\xi_2|(2\varepsilon + \frac{p}{2}(|\lambda\xi_1 + (1-\lambda)\xi_2|^2 - \varepsilon))}{(\varepsilon + |\lambda\xi_1 + (1-\lambda)\xi_2|^2)^{2-p/2}}|\xi_1 - \xi_2| \\ &\leq 4a|\lambda\xi_1 + (1-\lambda)\xi_2|^{p-1}|\xi_1 - \xi_2| \leq 4a\rho^{p-1}|\xi_1 - \xi_2|. \end{aligned}$$

The case $p \geq 2$:

$$\begin{aligned} |F_\varepsilon(\xi_1) - F_\varepsilon(\xi_2)| &\leq |\nabla F_\varepsilon(\lambda\xi_1 + (1-\lambda)\xi_2)||\xi_1 - \xi_2| \\ &\leq ap \cdot |\lambda\xi_1 + (1-\lambda)\xi_2|^{p-1}|\xi_1 - \xi_2| \leq ap\rho^{p-1}|\xi_1 - \xi_2|. \end{aligned}$$

Moreover, $F_\varepsilon \in C^\infty(\mathbb{R}^N)$ and satisfies the following inequalities:

$$(\nabla F_\varepsilon)(\xi)\cdot\xi - F_\varepsilon(\xi) \le a(p-1)|\xi|^p \quad \text{if } 0<p\le 1, \tag{3.4}$$

$$(\nabla F_\varepsilon)(\xi)\cdot\xi - F_\varepsilon(\xi) \ge a(p-1)|\xi|^p \quad \text{if } \ p>1. \tag{3.5}$$

Indeed, when $0<p\le 1$ we have:

$$\begin{aligned}
&(\nabla F_\varepsilon)(\xi)\cdot\xi - F_\varepsilon(\xi) = a\frac{p|\xi|^2-(\varepsilon+|\xi|^2)}{(\varepsilon+|\xi|^2)^{1-p/2}} = a\frac{(p-1)|\xi|^2-\varepsilon}{(\varepsilon+|\xi|^2)^{1-p/2}}\\
&= -a\frac{\varepsilon+(1-p)|\xi|^2}{(\varepsilon+|\xi|^2)^{1-p/2}} = -a\frac{\varepsilon+(1-p)|\xi|^2}{(\varepsilon+|\xi|^2)}\cdot(\varepsilon+|\xi|^2)^{p/2}\\
&\le -a\frac{(1-p)(\varepsilon+|\xi|^2)}{\varepsilon+|\xi|^2}\cdot(\varepsilon+|\xi|^2)^{p/2} \le -a(1-p)(\varepsilon+|\xi|^2)^{p/2} \le a(p-1)|\xi|^p.
\end{aligned}$$

If $1<p<2$ then:

$$\begin{aligned}
(\nabla F_\varepsilon)(\xi)\cdot\xi - F_\varepsilon(\xi) &= a\frac{(p-1)(\varepsilon+|\xi|^2)^2+3\varepsilon(2-p)|\xi|^2+\varepsilon^2(2-p)}{(\varepsilon+|\xi|^2)^{2-p/2}}\\
&\ge a(p-1)(\varepsilon+|\xi|^2)^{p/2} \ge a(p-1)|\xi|^p,
\end{aligned}$$

and finally, for $p\ge 2$ we have:

$$(\nabla F_\varepsilon)(\xi)\cdot\xi - F_\varepsilon(\xi) = ap|\xi|^{p-2}\xi\cdot\xi - a|\xi|^p = a(p-1)|\xi|^p.$$

For any $n\in\mathbb{N}$, let denote by:

$$\rho_n = \sup_{x\in\overline{\Omega}}\{|\nabla u_0^n|(x)\}. \tag{3.6}$$

Then, there exists $\delta\in(0,1)$ and a function $F_{n,\varepsilon}$, such that:

$$F_{n,\varepsilon}\in C^{2+\delta}(\mathbb{R}^N), \tag{3.7}$$

$$F_{n,\varepsilon}(\xi) = F_\varepsilon(\xi) \quad \text{if } \ \xi\in B_{\rho_n+1}(0), \tag{3.8}$$

$$F_{n,\varepsilon}(\xi) = \nu_n(1+|\xi|^2) \quad \text{if } \ |\xi|\ge\rho_n+2, \tag{3.9}$$

$$|F_{n,\varepsilon}(\xi)| \le \nu_n(1+|\xi|^2) \quad \text{for all } \ \xi\in\mathbb{R}^N, \tag{3.10}$$

where ν_n is a positive constant which depends only on ρ_n and p.
With $F_{n,\varepsilon}$ defined above we consider the problem:

$$\begin{cases}
\dfrac{\partial u}{\partial t} - \Delta u = F_{n,\varepsilon}(\nabla u) \text{ in } Q_T,\\
\dfrac{\partial u}{\partial \nu} = 0 \quad \text{ on } \ \Gamma_T,\\
u(0,\cdot) = u_0^n \ \text{ in } \ \Omega.
\end{cases} \tag{3.11}$$

Thanks to the regularity of u_0^n and to relations (3.1), (3.7), (3.8), (3.9) and (3.10) we can apply Theorem V.7.4 in [20] to the problem (3.11). Thus, there exists

$u^{n,\varepsilon} \in C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$,$\alpha \in (0,1)$, the unique solution of problem (3.11). For any $(t,x) \in Q_T$ let denote by:

$$f_{n,\varepsilon}(t,x) = F_{n,\varepsilon}(\nabla u^{n,\varepsilon})(t,x). \tag{3.12}$$

Then, thanks to the regularity of $F_{n,\varepsilon}$ and $u^{n,\varepsilon}$ it follows that: $f_{n,\varepsilon} \in C^{\frac{1+\alpha}{2},1+\alpha}(\overline{Q_T})$ and $u^{n,\varepsilon}$ verifies:

$$\begin{cases} \dfrac{\partial u^{n,\varepsilon}}{\partial t} - \Delta u^{n,\varepsilon} = f_{n,\varepsilon} \text{ in } Q_T, \\ \dfrac{\partial u^{n,\varepsilon}}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_T, \\ u^{n,\varepsilon}(0,\cdot) = u_0^n \text{ in } \quad \Omega. \end{cases} \tag{3.13}$$

Applying Theorem III.12.2 in [20], on the local regularity of solution for parabolic problem of (3.13) type, we get:

$$u^{n,\varepsilon} \in C_{loc}^{\frac{3+\alpha}{2},3+\alpha}(Q_T) \cap C^{1+\alpha/2,2+\alpha}(\overline{Q_T}).$$

In the sequel, we show that, for $\varepsilon \in (0,1)$,

$$|\nabla u^{n,\varepsilon}(t,x)| \leq \rho_n, \quad \forall\ (t,x) \in Q_T, \tag{3.14}$$

where ρ_n is given by (3.6). For this, we will use the Bernstein technique. First we introduce the parabolic operator $\mathcal{L}$ defined on $C^{0,1}(\overline{Q_T}) \cap C^{1,2}(Q_T)$ by:

$$\mathcal{L}(v) = \frac{\partial v}{\partial t} - \Delta v + b(t,x) \cdot \nabla v,$$

where $b \in [L^\infty(Q_T)]^N$ is given by:

$$b(t,x) = -(\nabla F_{n,\varepsilon})(\nabla u^{n,\varepsilon})(t,x) \text{ in } Q_T.$$

Setting $w = |\nabla u^{n,\varepsilon}|^2$, then $w \in C^{0,1}(\overline{Q_T}) \cap C^{1,2}(Q_T)$ and verifies:

$$\mathcal{L}(w) = -2 \sum_{i,j=1}^{N} \left(\frac{\partial^2 u^{n,\varepsilon}}{\partial x_i \partial x_j} \right)^2 \leq 0,$$

hence, thanks to Lemma 2.3 and to relations (3.1) and (3.6) we have:

$$\frac{\partial w}{\partial \nu} \leq 0 \text{ on } \Gamma_T \text{ and } \quad w(0,x) \leq \rho_n^2 \text{ in } \quad \Omega.$$

Then, by the Comparison Principle (Lemma 2.4), we obtain: $w \leq \rho_n^2$ in Q_T and relation (3.14) is proved.

Combining (3.8), (3.11) and (3.14), we finally obtain that: $u^{n,\varepsilon} \in C_{loc}^{\frac{3+\alpha}{2},3+\alpha}(Q_T) \cap C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$ is the solution of the initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = F_\varepsilon(\nabla u) \text{ in } Q_T, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{ on } \Gamma_T, \\ u(0,\cdot) = u_0^n \text{ in } \Omega. \end{cases} \tag{3.15}$$

Moreover, we notice that, in (3.15), F_ε is independent of n.

Second step: "Estimates for $u^{n,\varepsilon}$".
For $\varepsilon > 0$ and n a positive entire let set:

$$m_{n,\varepsilon} = m(0) + \frac{1}{n} - a\varepsilon^{p/2} \cdot T \tag{3.16}$$

and

$$M_{n,\varepsilon} = M(0) + \frac{2}{n} + a\varepsilon^{p/2} \cdot T. \tag{3.17}$$

The next proposition gives some estimates of $u^{n,\varepsilon}$ which will allow us to pass to the limits in (3.15), as ε tends to 0:

Proposition 3.1 *For all $p \in (0, +\infty)$, the solution $u^{n,\varepsilon} \in C_{loc}^{\frac{3+\alpha}{2},3+\alpha}(Q_T) \cap C^{1+\alpha/2,2+\alpha}(\overline{Q_T})$ of problem (3.15) satisfies:*

$$m_{n,\varepsilon} \le u^{n,\varepsilon} \le M_{n,\varepsilon} \quad \text{in} \quad Q_T, \tag{3.18}$$

$$\|\nabla u^{n,\varepsilon}(t)\|_\infty \le \left(\frac{1}{2}\right)^{1/2} (M_{n,\varepsilon} - m_{n,\varepsilon} + \frac{1}{n}) \cdot t^{-\frac{1}{2}} \text{ for all } t \in (0,T), \tag{3.19}$$

and, if $p \neq 1$:

$$\|\nabla u^{n,\varepsilon}(t)\|_\infty \le \left(\frac{\max\{p,2\}}{ap|1-p|}\right)^{1/p} (M_{n,\varepsilon} - m_{n,\varepsilon} + \frac{1}{n})^{1/p} \cdot t^{-1/p} \text{ for all } t \in (0,T). \tag{3.20}$$

Proof: The two inequalities in (3.18) are simple consequences of Lemma 2.4.
Instead, to prove (3.19) and (3.20) we will use the Bernstein technique and the proof is similar to that given in [9],[17] and [22]. Let denote by w the function defined on Q_T by:

$$w = \frac{|\nabla u^{n,\varepsilon}|^2}{\theta(u^{n,\varepsilon})}. \tag{3.21}$$

where θ is a strict positive function of $C^2([m_{n,\varepsilon}, M_{n,\varepsilon}])$ class, which will be chosen later according to the exponent p.
Then, thanks to the regularity of function $u^{n,\varepsilon}$ we have:

$$w \in C^{0,1}(\overline{Q_T}) \cap C^{1,2}(Q_T).$$

Moreover:

$$\begin{aligned}\frac{\partial w}{\partial \nu} &= \frac{1}{[\theta(u^{n,\varepsilon})]^2}\left(\frac{\partial|\nabla u^{n,\varepsilon}|^2}{\partial \nu}\theta(u^{n,\varepsilon}) - |\nabla u^{n,\varepsilon}|^2 \cdot \theta'(u^{n,\varepsilon})\frac{\partial u^{n,\varepsilon}}{\partial \nu}\right)\\ &= \frac{1}{[\theta(u^{n,\varepsilon})]} \cdot \frac{\partial|\nabla u^{n,\varepsilon}|^2}{\partial \nu} \quad \text{on } \Gamma_T.\end{aligned}$$

Since θ is a positive function, this last relation and Lemma 2.3 imply:

$$\frac{\partial w}{\partial \nu} \leq 0 \text{ on } \Gamma_T. \tag{3.22}$$

Denote by $\mathcal{N}$ the semi-linear parabolic operator defined on $C^{0,1}(\overline{Q_T}) \cap C^{1,2}(Q_T)$ by:

$$\mathcal{N}(v) = \frac{\partial v}{\partial t} - \Delta v - b(t,x)\cdot \nabla v - c(t,x)v^2 - d(t,x)v^{1+p/2}$$

where

$$b(t,x) = (\nabla F_\varepsilon)(\nabla u^{n,\varepsilon})(t,x) + \frac{2\theta'(u^{n,\varepsilon})\nabla u^{n,\varepsilon}(t,x)}{\theta(u^{n,\varepsilon})(t,x)},$$

$$c(t,x) = \theta''(u^{n,\varepsilon})(t,x),$$

and

$$d(t,x) = a(p-1)\theta^{\frac{p-2}{2}}(u^{n,\varepsilon})(t,x)\theta'(u^{n,\varepsilon})(t,x). \tag{3.23}$$

The function w being introduced by (3.21) we have:

$$\begin{aligned}\mathcal{N}(w) = -\frac{2}{\theta(u^{n,\varepsilon})}\sum_{i,j=1}^{N}\left(\frac{\partial^2 u^{n,\varepsilon}}{\partial x_i \partial x_j}\right)^2 + \frac{\theta'(u^{n,\varepsilon})}{\theta^2(u^{n,\varepsilon})}[(\nabla F_\varepsilon)(\nabla u^{n,\varepsilon})\cdot\nabla u^{n,\varepsilon}\\ -F_\varepsilon(\nabla u^{n,\varepsilon}) - a(p-1)|\nabla u^{n,\varepsilon}|^p]|\nabla u^{n,\varepsilon}|^2.\end{aligned} \tag{3.24}$$

To prove (3.19) we will distinguish between the two cases below.
i) The case $0 < p \leq 1$. We take θ in (3.21) as follows:

$$\theta(\xi) = \frac{1}{2}(M_{n,\varepsilon} - m_{n,\varepsilon} + \frac{1}{n})^2 - \frac{1}{2}(M_{n,\varepsilon} - \xi)^2, \quad \xi \in [m_{n,\varepsilon}, M_{n,\varepsilon}],$$

where $m_{n,\varepsilon}$ and $M_{n,\varepsilon}$ are defined by (3.16) and (3.17). So θ verifies:

$$\theta(\xi) \geq \frac{1}{2n^2}, \qquad \theta'(\xi) = M_{n,\varepsilon} - \xi, \qquad \theta''(u) = -1,$$

and we deduce that:

$$\theta'(u^{n,\varepsilon}) \geq 0$$

and

$$d(t,x) = a(p-1)\theta^{\frac{p-2}{2}}(u^{n,\varepsilon})(t,x)\theta'(u^{n,\varepsilon})(t,x) \leq 0.$$

Combining these last points with (3.4) and (3.24) it follows that:

$$\mathcal{N}(w) \le 0. \tag{3.25}$$

Taking into account (3.6) and (3.13) we have:

$$w(0) = \frac{|\nabla u_0^n|^2}{\theta(u_0^n)} \le 2\rho_n^2 n^2.$$

So, for n a fixed entire, choose $\eta > 0$ such that:

$$w(0) \le 2\rho_n^2 n^2 \le \frac{1}{\eta}, \tag{3.26}$$

and denote by v the function defined on Q_T by:

$$v(t,x) = (t+\eta)^{-1}.$$

Since $a > 0$ and $p \in (0,1)$ we have:

$$\mathcal{N}(v) = -d(t,x) \cdot (t+\eta)^{-(1+p/2)} \ge 0. \tag{3.27}$$

So, recalling (3.22),(3.25), (3.26), (3.27) and Lemma 2.4 we get:

$$w(t,x) \le (t+\eta)^{-1} < t^{-1} \text{ for all } (t,x) \in Q_T,$$

and we deduce that (3.19) holds for $p \in (0,1]$.

ii) The case $p > 1$. In (3.21) we consider the function θ defined by:

$$\theta(\xi) = \frac{1}{2}(M_{n,\varepsilon} - m_{n,\varepsilon} + \frac{1}{n})^2 - \frac{1}{2}(\xi - m_{n,\varepsilon})^2, \quad \xi \in [m_{n,\varepsilon}, M_{n,\varepsilon}],$$

then θ satisfies:

$$\theta(\xi) \ge \frac{1}{2n^2}, \qquad \theta'(\xi) = m_{n,\varepsilon} - \xi, \qquad \theta''(\xi) = -1,$$

and we deduce that:

$$\theta'(u^{n,\varepsilon}) \le 0,$$

and

$$d(t,x) = a(p-1)\theta^{\frac{p-2}{2}}(u^{n,\varepsilon})(t,x)\theta'(u^{n,\varepsilon})(t,x) \le 0.$$

Combining these last points with (3.5) and (3.24) it follows that:

$$\mathcal{N}(w) \le 0.$$

As previously, we can prove (3.19) for the case $p \ge 1$ by comparing w and v.

To prove (3.20) we will distinguish among three cases:
$i)$ The case $0 < p < 1$. In (3.21), we consider the following function:

$$\theta(\xi) = (\frac{2}{ap(1-p)})^{2/p}(M_{n,\varepsilon} - m_{n,\varepsilon} + \frac{1}{n})^{\frac{2-p}{p}} \cdot (\xi - m_{n,\varepsilon} + \frac{1}{n}),\ \xi \in [m_{n,\varepsilon}, M_{n,\varepsilon}].$$

Thus

$$\theta(\xi) \geq \left[\frac{2}{anp(1-p)}\right]^{2/p}, \quad \theta'(\xi) \geq 0 \quad \text{and} \quad \theta''(\xi) = 0,\ \xi \in [m_{n,\varepsilon}, M_{n,\varepsilon}]. \tag{3.28}$$

w being given by (3.21), thanks to relations (3.4) and (3.24) we obtain:

$$\mathcal{N}(w) \leq 0 \tag{3.29}$$

Taking into account (3.11) and (3.28), we can choose $\eta > 0$ such that:

$$w(0) = \frac{|\nabla u_0^n|^2}{\theta(u_0^n)} \leq 2\rho_n^2 \left[\frac{anp(1-p)}{2}\right]^{2/p} < \frac{1}{\eta^{2/p}}, \tag{3.30}$$

where ρ_n is given by (3.6).
Let v be a function defined on Q_T by:

$$v(t,x) = (t+\eta)^{-2/p}.$$

With d given by (3.23), and θ being chosen as above we have:

$$d(t,x) = -\frac{2}{p}\left(\frac{M_{n,\varepsilon} - m_{n,\varepsilon} + \frac{1}{n}}{u - m_{n,\varepsilon} + \frac{1}{n}}\right)^{\frac{2-p}{2}} \leq -\frac{2}{p}$$

And we deduce that:

$$\mathcal{N}(v) = (-d - 2/p)(t+\eta)^{-\frac{p+2}{2}} \geq 0 \tag{3.31}$$

Combining relations (3.22),(3.29),(3.30),(3.31), and Lemma 2.4 we get:

$$w(t,x) \leq (t+\eta)^{2/p} < t^{-2/p} \text{ for all } t > 0,$$

and we deduce (3.20), for $0 < p \leq 1$.
$ii)$ The case $1 < p < 2$. In (3.21), we choose the following function θ:

$$\theta(\xi) = \left[\frac{2}{ap(p-1)}\right]^{2/p}(M_{n,\varepsilon} - m_{n,\varepsilon} + \frac{1}{n})^{\frac{2-p}{p}}(M_{n,\varepsilon} - \xi + \frac{1}{n}),\ \xi \in [m_{n,\varepsilon}, M_{n,\varepsilon}]$$

Thus:

$$\theta(\xi) \geq \left[\frac{2}{anp(p-1)}\right]^{2/p}, \quad \theta'(\xi) \leq 0 \ \text{ and } \ \theta''(\xi) = 0,\ \xi \in [m_{n,\varepsilon}, M_{n,\varepsilon}]. \tag{3.32}$$

and we get (3.20) as previously.
iii) The case $p \geq 2$. This time we prove (3.20) in the two cases above, by taking:

$$\theta(\xi) = \Big[\frac{1}{a(p-1)}(M_{n,\varepsilon} - \xi + \frac{1}{n})\Big]^{2/p}, \ \xi \in [m_{n,\varepsilon}, M_{n,\varepsilon}].$$

□

We came back to the problem (3.15) and we notice that

$$\varepsilon \mapsto F_\varepsilon(\xi) \text{ is a nondecreasing function for } \ 0 < p \leq 1$$

and

$$\varepsilon \to F_\varepsilon(\xi) \text{ is a decreasing function for } \ p > 1.$$

Then, thanks to relations (3.3) and (3.18) we can apply Lemma 2.4 and we obtain that the set $(u^{n,\varepsilon})_{\varepsilon>0}$ is bounded and monotone with respect to ε, and consequently, there exists $u^n \in L^\infty(Q_T)$ such that

$$u^{n,\varepsilon} \nearrow u^n \text{ in } Q_T \ \text{ as } \varepsilon \searrow 0, \ \text{ if } \ 0 < p \leq 1$$

and

$$u^{n,\varepsilon} \searrow u^n \text{ in } Q_T \ \text{ as } \varepsilon \searrow 0, \ \text{ if } \ p > 1.$$

Moreover, from relations (3.1) and (3.18), the hypotheses of Theorem V.7.2 in [20] are satisfied and we deduce that the solutions $u^{n,\varepsilon}$ of (3.15) verify:

$$\|u^{n,\varepsilon}\|_{C^{\frac{1+\delta}{2},1+\delta}(Q_T)} \leq C \tag{3.33}$$

where $\delta \in (0,1)$ and C are two positive constants which depend only on $m, M, \|u_0^n\|_\Omega^{(2)}$ and Ω. Thus, we deduce that for all n, the set $\{u^{n,\varepsilon}, 0 < \varepsilon < 1\}$ is bounded in $C^{\frac{1+\delta}{2},1+\delta}(\overline{Q_T})$. Let be $f_{n,\varepsilon}$ the function given by (3.12), then, thanks to the regularity of F_ε and to (3.33), the set $\{f_{n,\varepsilon}, 0 < \varepsilon < 1\}$ is bounded in $C^{\delta/2,\delta}(\overline{Q_T})$. Since $u^{n,\varepsilon} \in C^{\frac{1+\delta}{2},1+\delta}(\overline{Q_T})$ is the solution of problem (3.15), the hypotheses of Theorem IV.5.3 in [20] on the regularity in Hölder spaces of solutions for parabolic equations, are verified and therefore we get:

$$u^{n,\varepsilon} \in C^{1+\delta/2,2+\delta}(\overline{Q_T}),$$

moreover, there exists a constant $C > 0$, not depending on $\varepsilon \in (0,1)$, such that:

$$\|u^{n,\varepsilon}\|_{C^{1+\delta/2,2+\delta}(\overline{Q_T})} \leq C(\|u_0^n\|_{C^{2+\delta}(\overline{\Omega})} + \|f_{n,\varepsilon}\|_{C^{\delta/2,\delta}(\overline{Q_T})} \leq C_n \tag{3.34}$$

Thus, the set $\{u^{n,\varepsilon}, 0 < \varepsilon < 1\}$ is bounded in $C^{1+\delta/2,2+\delta}(\overline{Q_T})$. Since for any $0 \leq \nu < \delta$

$$C^{1+\delta/2,2+\delta}(\overline{Q_T}) \hookrightarrow C^{1+\nu/2,2+\nu}(\overline{Q_T}),$$

with compact embedding, we deduce that $\{u^{n,\varepsilon}, 0 < \varepsilon < 1\}$ is a precompact set in $C^{1+\nu/2,2+\nu}(\overline{Q_T})$ and it follows that, "to a subsequence" we have:

$$u^{n,\varepsilon} \to u^n \quad \text{in } C^{1+\nu/2,2+\nu}(\overline{Q_T}) \text{ as } \varepsilon \searrow 0 \tag{3.35}$$

On the other hand, for all $\xi \in \mathbb{R}^N$:

$$F_\varepsilon(\xi) \to a|\xi|^p \text{ as } \varepsilon \searrow 0,$$

So, we can pass to the limit in (3.15), as $\varepsilon \searrow 0$, and we obtain that $u^n \in C^{1+\nu/2,2+\nu}(\overline{Q_T})$ is a solution of the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial u^n}{\partial t} - \Delta u^n = a|\nabla u^n|^p \quad \text{in} \quad Q_T, \\ \dfrac{\partial u^n}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_T, \\ u^n(0,x) = u_0^n(x) \quad \text{in} \quad \Omega. \end{cases} \tag{3.36}$$

Applying the Comparison Principle, [Theorem 1 in [19]], we get also that this solution is unique in $C^{1,2}(\overline{Q_T})$.

Third step: "Estimates for u^n ".

The aim of the following proposition is to prove that $(u^n)_n$ satisfies also the estimates (3.18), (3.19) and (3.20) for $\varepsilon = 0$, and is bounded in a Hölder space.

Proposition 3.2 *The solution $u^n \in C^{1+\nu/2,2+\nu}(\overline{Q_T})$ of problem (3.36) satisfies the following properties:*

$$m(0) + \frac{1}{n} \le u^n(t,x) \le M(0) + \frac{2}{n}, \tag{3.37}$$

$$\|\nabla u^n(t)\|_\infty \le \left(\frac{1}{2}\right)^{1/2} (M(0) - m(0) + \frac{2}{n}) \cdot t^{-\frac{1}{2}}, \ \textit{for all } t \in (0,T), \tag{3.38}$$

and, if $p \neq 1$ then:

$$\|\nabla u^n(t)\|_\infty \le \left(\frac{\max\{p,2\}}{ap|1-p|}\right)^{1/p} (M(0) - m(0) + \frac{2}{n})^{1/p} \cdot t^{-1/p}, \ \textit{for all } t \in (0,T). \tag{3.39}$$

Moreover, there exists $\delta \in (0,1)$ such that, for all $\tau \in (0,T)$:

$$\textit{the sequence } (u^n)_n \textit{ is bounded in } \ C^{1+\delta/2,2+\delta}(\overline{Q_{\tau,T}}). \tag{3.40}$$

(This bound depends only on $\tau, \Omega, p, m(0)$ and $M(0)$.)

Proof: Relations (3.37), (3.38) and (3.39) are direct consequences of (3.18), (3.19), (3.20) and (3.35). In order to prove (3.40) we denote by f_n the function defined on Q_T by:

$$f_n(t,x) = a|\nabla u^n|^p(t,x).$$

Then $u^n \in C^{1+\frac{\nu}{2},2+\nu}(\overline{Q_T})$ is the solution of the following problem:

$$\begin{cases} \dfrac{\partial u^n}{\partial t} - \Delta u^n = f_n \ \text{ in } Q_T, \\ \dfrac{\partial u^n}{\partial \nu} = 0 \quad \text{on } \Gamma_T, \\ u^n(0,\cdot) = u_0^n \ \text{ in } \Omega. \end{cases} \tag{3.41}$$

Consider $\tau \in (0,T)$. Thanks to relation (3.38), $f_n \in L^\infty(Q_{\tau,T})$ and:

$$\|f_n\|_{L^\infty(Q_{\tau,T})} \leq \frac{a}{2^{p/2}}(M(0) - m(0) + 2)^p \tau^{-p/2}, \quad \forall\, n \in \mathbb{N} \tag{3.42}$$

Consequently, the sequence $(f_n)_{n\geq 0}$ is uniformly bounded in $L^\infty(Q_{\tau,T})$.
In the sequel, we decompose the problem (3.41) into two parts.
On the one hand, we denote by v^n the solution of the heat equation on $Q_{\tau/3,T}$:

$$\begin{cases} \dfrac{\partial v^n}{\partial t} - \Delta v^n = 0 \quad \text{in } \ Q_{\tau/3,T}, \\ \dfrac{\partial v^n}{\partial \nu} = 0 \quad \text{on } \ \Gamma_{\tau/3,T}, \\ v^n(\tau/3,x) = u^n(\tau/3,x) \quad \text{in } \ \Omega. \end{cases} \tag{3.43}$$

Thanks to the regularity effect of the heat equation it follows that:

$$v^n \in C^\infty(\overline{Q_{2\tau/3,T}}) \tag{3.44}$$

and from Lemma 2.6 and relations (2.15) and (3.37), for all $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^N$ we have:

$$\|D_x^\alpha D_t^l v^n\|_{\infty,Q_{2\tau/3,T}} \leq C(T,\Omega)(M(0) + m(0) + 1)\tau^{-(\frac{|\alpha|}{2}+l)}. \tag{3.45}$$

Next, we denote by w^n the solution of the problem:

$$\begin{cases} \dfrac{\partial w^n}{\partial t} - \Delta w^n = f_n(t,x) \ \text{ in } Q_{\tau/3,T}, \\ \dfrac{\partial w^n}{\partial \nu} = 0 \quad \text{on } \Gamma_{\tau/3,T}, \\ w^n(\tau/3,\cdot) = 0 \ \text{ in } \Omega. \end{cases} \tag{3.46}$$

Taking into account (3.35), we have $f_n \in C^{\frac{1+\nu}{2},1+\nu}(\overline{Q_T})$ and we deduce that $w^n \in C^{1+\nu/2,2+\nu}(\overline{Q_{\tau/3,T}})$. Since $f_n \in L^\infty(Q_{\tau/3,T})$, we have in particular $f_n \in L^q(Q_{\tau/3,T})$ for all $q > 1$. Thus, we can apply Theorem 7.20 in [21], on the regularity of parabolic

solutions in L^q spaces, and we get that, there exists a constant $C > 0$, independent on n, such that:

$$\|D_x^2 w^n\|_{q,Q_{\frac{\tau}{3},T}} + \|D_t w^n\|_{q,Q_{\frac{\tau}{3},T}} \leq C\|f_n\|_{q,Q_{\frac{\tau}{3},T}} \leq C|Q_{\frac{\tau}{3},T}|^{1/q}\|f_n\|_{\infty,Q_{\frac{\tau}{3},T}}. \quad (3.47)$$

Combining (3.42) with (3.47) we get:

$$\|D_t w^n\|_{q,Q_{\frac{\tau}{3},T}} + \|D_x^2 w^n\|_{q,Q_{\frac{\tau}{3},T}} \leq C(M(0), m(0), p, \tau, T, \Omega). \quad (3.48)$$

Since $u^n = v^n + w^n$, from (3.45) and (3.48) we get on the one hand:

$$\|D_t u^n\|_{q,Q_{2\tau/3,T}} + \|D_x^2 u^n\|_{q,Q_{2\tau/3,T}} \leq C(M(0), m(0), p, q, \tau, T, \Omega). \quad (3.49)$$

On the other hand relations (3.37) and (3.38) yield:

$$\|u^n\|_{\infty,Q_{2\tau/3,T}} \leq C_1(M(0), m(0)) \quad (3.50)$$

and

$$\|D_x u^n\|_{\infty,Q_{2\tau/3,T}} \leq C_2(M(0), m(0), p, \tau). \quad (3.51)$$

So, combining (3.49), (3.50) and (3.51) we get:

$$\|u^n\|_{W_q^{1,2}(Q_{2\tau/3,T})} \leq C(M(0), m(0), p, q, \tau, T, \Omega) \text{ for all } n \in \mathbb{N} \text{ and } q > 1.$$

We choose $q > N + 2$, then, applying Lemma II.3.3 in [20] (on the embedding of Sobolev spaces into Hölder spaces), we deduce that, for any β satisfying $0 < \beta < 1 - \frac{N+2}{q}$, there exists a constant $C > 0$ such that:

$$\|\nabla u^n\|_{C^{\beta/2,\beta}(\overline{Q_{2\tau/3,T}})} \leq C(q, \beta, \tau, T, \Omega)\|u^n\|_{W_q^{1,2}(Q_{2\tau/3,T})}.$$

Since the sequence $(u_n)_n$ is bounded in $W_q^{1,2}(Q_{2\tau/3,T})$, we deduce that $(|\nabla u_n|)_n$ is bounded in $C^{\beta/2,\beta}(\overline{Q_{2\tau/3,T}})$. Consequently the sequence $(f_n = a|\nabla u_n|^p)_n$ is uniformly bounded in $C^{\delta/2,\delta}(Q_{2\tau/3,T})$, where $\delta = \delta(\beta, p)$.

We came back to problems (3.41), (3.43) and (3.46) in $Q_{2\tau/3,T}$. By reiterating the process above we get, thanks to Theorem IV.5.3 in [20], that:
i) $w^n \in C^{1+\delta/2,2+\delta}(\overline{Q_{2\tau/3,T}})$ and there exists a constant $C > 0$, independent on n such that:

$$\|w^n\|_{C^{1+\delta/2,2+\delta}(\overline{Q_{\frac{2\tau}{3},T}})} \leq C\|f_n||_{C^{\delta/2,\delta}(\overline{Q_{\frac{2\tau}{3},T}})} \leq C(m(0), M(0), p, N, \tau, T, \Omega). \quad (3.52)$$

ii) v^n satisfies relation (3.45) on $Q_{\tau,T}$.

Thus, recalling (3.45) and (3.52) we obtain that $u^n \in C^{1+\delta/2,2+\delta}(\overline{Q_{\tau,T}})$ and:

$$\|u^n\|_{C^{1+\delta/2,2+\delta}(\overline{Q_{2\tau/3,T}})} \leq C(m(0), M(0), p, N, \tau, T, \Omega), \quad (3.53)$$

which ends the proof of Proposition 3.2. □

Four step: "Proof of the existence of solutions".
On the one hand, thanks to the Comparison Principle, [Theorem 1 in [19]], and to relations (3.1) and (3.37) the sequence $(u^n)_n$ is decreasing and uniformly bounded. Consequently, there exists $u \in L^\infty(Q_T)$ such that:

$$u^n \searrow u \quad \text{in} \quad Q_T. \tag{3.54}$$

On the other hand, by Proposition 3.2 we deduce that, for any $\tau \in (0, T)$, the sequence $(u^n)_{n\geq 1}$ is bounded in $C^{1+\delta/2,2+\delta}(\overline{Q_{\tau,T}})$. Since for all $\nu \in (0, \delta)$:

$$C^{1+\delta/2,2+\delta}(\overline{Q_{\tau,T}}) \hookrightarrow C^{1+\nu/2,2+\nu}(\overline{Q_{\tau,T}})$$

with compact embedding, "to a subsequence", we have:

$$u^n \to u \;\; \text{in} \;\; C^{1+\nu/2,2+\nu}(\overline{Q_{\tau,T}}) \;\; \text{as} \;\; n \to \infty. \tag{3.55}$$

Hence, $u \in C^{1+\nu/2,2+\nu}(\overline{Q_{\tau,T}})$ and thanks to relations (3.54) and (3.55) we may let $t \to \infty$ in the first and the second equation of problem (3.36) and we obtain that, for all $\tau \in (0, T)$, u satisfies:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = a|\nabla u|^p \;\; \text{in } Q_{\tau,T}, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_{\tau,T}. \end{cases} \tag{3.56}$$

Moreover, passing to limits in (3.38) and (3.39), as n tends to ∞, we get (1.6) and (1.7). The relations (1.4) and (1.5) are direct consequences of Lemma 2.4.
So, we have to identify the initial data μ_0. For $t \in (0, T)$, let denote by $v(t) = S(t)\mu_0$ and $v^n = S(t)u_0^n$, where $(S(t))_{t\geq 0}$ is the heat semigroup in $L^q(\Omega), q \geq 1$, for the homogeneous Neumann boundary value problem. Then, by the Comparison Principle [Lemma 2.4], for $n \in \mathbb{N}$ we have:

$$v^n \leq u^n \;\; \text{in} \;\; Q_T.$$

Using (3.54), we may let $n \to \infty$ in the above inequality and we obtain:

$$v \leq u \;\; \text{in} \;\; Q_T. \tag{3.57}$$

Since $v \in C(\overline{Q_T})$, it follows that:

$$\mu_0(x_0) = \lim_{\substack{(t,x)\to(0,x_0)\\(t,x)\in Q_T}} v(t, x) \leq \liminf_{\substack{(t,x)\to(0,x_0)\\(t,x)\in Q_T}} u(t, x), \tag{3.58}$$

for any $x_0 \in \Omega$. Furthermore for $n \in \mathbb{N}$ we have:

$$u \leq u^n \;\; \text{in} \;\; Q_T,$$

then:

$$\limsup_{\substack{(t,x)\to(0,x_0)\\(t,x)\in Q_T}} u(t,x) \leq \limsup_{\substack{(t,x)\to(0,x_0)\\(t,x)\in Q_T}} u^n(t,x) = u_0^n(x_0).$$

Since $(u_0^n)_n$ is a decreasing sequence and converges to μ_0 we can pass to the limits in the above inequality and we get

$$\limsup_{\substack{(t,x)\to(0,x_0)\\(t,x)\in Q_T}} u(t,x) \leq \mu_0(x_0). \tag{3.59}$$

Combining (3.58), (3.59) and the fact that x_0 is anywhere in Ω we deduce that $u \in C(\overline{Q_T}) \cap C^{1+\nu/2,2+\nu}(\overline{Q_{\tau,T}})$ is a classical solution of the problem (1.1). Which end the existence proof of solutions of problem (1.1), for $a > 0$.

Fifth step: "Uniqueness of the solution".

The uniqueness is a direct consequence of the following lemma:

Lemma 3.3 *Let $a > 0$, $p > 1$, $\Omega \subset \mathbb{R}^N$ a bounded and convex open set with smooth boundary. Let $\mu_0 \in C(\overline{\Omega})$ and $u \in C(\overline{Q_T}) \cap C^{1+\nu/2,2+\nu}(\overline{Q_{\tau,T}})$ the solution of problem (1.1) found above. Consider $w_0 \in C(\overline{\Omega})$ and $w \in C(\overline{Q_T}) \cap C^{1+\nu/2,2+\nu}(\overline{Q_{\tau,T}})$ a function satisfying:*

$$\begin{cases} \dfrac{\partial w}{\partial t} - \Delta w \leq a|\nabla w|^p \ (\ \geq a|\nabla w|^p\ ) \ \ \text{in} \ \ Q_T, \\ \dfrac{\partial w}{\partial \nu} \leq 0 \ (\ \geq 0\ ) \ \ \text{on} \ \ \Gamma_T, \\ w(0,\cdot) = w_0 \leq \mu_0 \ (\text{resp. } w_0 \geq \mu_0\ ) \ \ \text{in} \ \ \Omega. \end{cases} \tag{3.60}$$

Then:

$$w \leq u \ (\text{resp. } w \geq u\ ) \ \ \text{in} \ \ Q_T.$$

Proof: An analogous result for the whole space $\mathbb{R}^N$ can be found in [17] [Lemma 7] and our proof follows the same arguments.

We suppose first that Ω is a bounded and convex open set which contains the origin and $w_0 \leq \mu_0$.

Consider two real numbers $\varepsilon > 0$ and $A > 0$, and denote by z the function:

$$z(t,x) = w(t,x) - u(t,x) - At^q - \varepsilon(1+|x|^2)^{\frac{1}{2}}, \tag{3.61}$$

where $q = \min\{1, \frac{1}{p}\}$. Then:

$$z \in C(\overline{Q_T}) \cap C^{1,2}((0,T]\times\overline{\Omega}) \ \ \text{and} \ \ z(0,x) \leq 0 \ \ \text{for all} \ \ x \in \Omega.$$

Thanks to Lemma 2.3 and to hypothesis (3.60) we have:

$$\frac{\partial z}{\partial \nu}(t,x) = \frac{\partial w}{\partial \nu}(t,x) - \frac{\partial u}{\partial \nu}(t,x) - \frac{\varepsilon \cdot x \cdot \nu}{\sqrt{1+|x|^2}} < 0 \ \ \text{on} \ \ \Gamma_T. \tag{3.62}$$

We claim that

$$z(t,x) \le 0 \quad \text{for all} \quad (t,x) \in Q_T, \tag{3.63}$$

Indeed, if z is positive anywhere in Q_T then z has a positive maximum in $(t_0, x_0) \in (0,T] \times \Omega$ since, if $(t_0, x_0) \in (0,T] \times \partial\Omega$, we have:

$$\frac{\partial z}{\partial \nu}(t_0, x_0) = \lim_{\lambda \nearrow 0} \frac{z(t_0, x_0 + \lambda\nu) - z(t_0, x_0)}{\lambda} \ge 0,$$

which contradicts relation (3.62).The rest of the proof is standard and follows the same ideas as the proof of Lemma 7 in [17]. So, it will be omitted.

In the general case,when Ω does not contain the origin it is enough to translate the problem on a domain which contains the origin, for example $\Omega_{x_0} = \Omega - x_0$ where $x_0 \in \Omega$. □

Remark 3.4 *The result of Lemma 3.3 is valid for all $a \in \mathbb{R}, a \neq 0$ an so, for any solution $u \in C(\overline{Q_T}) \cap C^{1+\delta/2,2+\delta}(\overline{Q_{\tau,T}})$ of problem (1.1).*

4 Proof of Theorem 1.3

Let β be a positive number satisfying:

$$\beta > \left(\frac{2p+1-N}{N}\right)_+ \tag{4.1}$$

and set:

$$\gamma = N(\beta+1) \quad \text{and} \quad \eta = N(\beta+1) - p. \tag{4.2}$$

Since β satisfies (4.1) we have:

$$\eta > p+1 \tag{4.3}$$

with this notations, we can state the following proposition which is the key argument in the proof of Theorem 1.3.

Proposition 4.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded and convex domain with smooth boundary and $\mu_0 \in C(\overline{\Omega})$. Let denote by u the solution of problem (1.1) whose existence was proved in Theorem 1.2. Then:*

i) The application $t \mapsto (1+t)(M(t) - m(t))^\gamma$ belongs to $L^1(0,+\infty)$,

ii) Denoting by y the function defined on $[0,+\infty)$ by:

$$y(t) = \int_t^\infty (s-t)(M(s) - m(s))^\gamma \, ds,$$

then $y \in W^{2,\infty}((0,+\infty))$ *and satisfies the following differential inequality:*

$$y'(t) + Cy(t)^{\alpha} \leq 0, \quad \forall\ t \in [0,+\infty), \tag{4.4}$$

where the positive constant depends only on $p, \beta, N, \Omega, (M(0) - m(0))$ *and*

$$\alpha = \frac{1+\eta}{2+\eta-p}. \tag{4.5}$$

Proof: The proof follows the same ideas as those of Lemma 3 in [7] and Lemma 12 in [8]. Setting:

$$T^* = \inf\{t > 0; |\nabla u(t)| \equiv 0\}$$

then T^* can be also defined by:

$$T^* = \inf\{t > 0; M(t) = m(t)\} = \inf\{t > 0; y(t) = 0\}. \tag{4.6}$$

First, if $\mu_0 \equiv c$ then $u \equiv c$, which implies $T^* = 0$ and Proposition 4.1 is achieved. We suppose that μ_0 is not constant, consequently $T^* \in (0,+\infty]$. Consider $T \in (0,T^*)$ and $t \in [0,T)$. Integrating the first equation of problem (1.1) on $(t,T) \times \Omega$, and using relations (1.2), (1.3), (1.4) and (1.5) we get:

$$|a| \int_t^T \int_{\Omega} |\nabla u(s,x)|^p \, dx \, ds \leq |\Omega|(M(t) - m(t))$$

Recalling (4.2) and (4.3) we have: $\gamma = N(\beta+1) = \eta + p$ and we deduce that:

$$\|\nabla u(s)\|_{\gamma}^{\gamma} \leq \|\nabla u(s)\|_{\infty}^{\eta} \cdot \||\nabla u(s)|^p\|_1.$$

Combining these two last inequalities we get:

$$\int_t^T \|\nabla u(s)\|_{\infty}^{-\eta} \cdot \|\nabla u(s)\|_{\gamma}^{\gamma} ds \leq \frac{|\Omega|}{|a|}(M(t) - m(t)) \tag{4.7}$$

We distinguish between the two cases below:
(i) The case $p \neq 1$. Thanks to relation (1.7), for all $s \in (t,T)$ we have:

$$\|\nabla u(s)\|_{\infty}^{-\eta} \geq C_1 (s-t)^{\eta/p} (M(t) - m(t))^{-\eta/p}, \tag{4.8}$$

where C_1 is a positive constant which depends only on $p > 0$.
Applying Lemma 2.2 we have:

$$M(s) - m(s) \leq C_2 \|\nabla u(s)\|_{\gamma} \tag{4.9}$$

And combining (4.7),(4.8) and (4.9) we get:

$$\int_t^T (s-t)^{\eta/p}(M(s)-m(s))^\gamma\, ds \le C_3(M(t)-m(t))^{\gamma/p} \text{ for all } t\in[0,T], \tag{4.10}$$

where the constant C_3 depends only on N, a, p, β, η and Ω.
We can pass to the limit in (4.10), as $T \nearrow T^*$, and we obtain:

$$\int_t^{T^*} (s-t)^{\eta/p}(M(s)-m(s))^\gamma\, ds \le C_3(M(t)-m(t))^{\gamma/p} \text{ for all } t\in[0,T^*) \tag{4.11}$$

Let fix $\delta\in(0,T^*)$. Using (1.4),(1.5),(4.3) and (4.11) we get:

$$\begin{aligned}
&\int_0^\infty (1+s)(M(s)-m(s))^\gamma\, ds = \\
&\quad = \int_0^\delta (1+s)(M(s)-m(s))^\gamma\, ds + \int_\delta^{T^*} (1+s)(M(s)-m(s))^\gamma\, ds \\
&\quad \le (1+\delta)[\delta(M(0)-m(0))^\gamma + \delta^{-\eta/p}\int_\delta^{T^*} s^{\eta/p}(M(s)-m(s))^\gamma\, ds] \\
&\quad \le (1+\delta)[\delta(M(0)-m(0))^\gamma + \delta^{-\eta/p}\int_0^{T^*} s^{\eta/p}(M(s)-m(s))^\gamma\, ds].
\end{aligned}$$

Once again, using (1.4), (1.5) and (4.11) (which is, in particular, valid for $t=0$), this last integral is finite. Consequently:

$$t\mapsto (1+t)(M(t)-m(t))^\gamma \in L^1((0,+\infty)). \tag{4.12}$$

And we deduce that the function y is well defined on $[0,+\infty)$ and belongs to $W^{2,\infty}((0,+\infty))$. Indeed, for $t>0$, we have:

$$y'(t) = -\int_t^{+\infty} (M(s)-m(s))^\gamma\, ds \tag{4.13}$$

and

$$y''(t) = (M(t)-m(t))^\gamma. \tag{4.14}$$

Using Hölder inequality and (4.3) we deduce that the function y verifies:

$$\begin{aligned}(y(t))^{\eta/p} &= (\int_t^{\infty}(s-t)(M(s)-m(s))^{\gamma}\,ds)^{\eta/p}\\ &\leq \Big[\int_t^{T^*}(s-t)^{\eta/p}(M(s)-m(s))^{\gamma}\,ds\Big]\Big[\int_t^{T^*}(M(s)-m(s))^{\gamma}\,ds\Big]^{\eta/p-1}\end{aligned}$$

Combining this last inequality with (4.10) we get:

$$y(t)^{\eta/p} \leq C\cdot y''(t)^{1/p}(-y'(t))^{\eta/p-1}, \tag{4.15}$$

which yields:

$$y(t)^{\eta} \leq C\cdot y''(t)\cdot(-y'(t))^{\eta-p}, \quad \forall\ t\in[0,T^*), \tag{4.16}$$

Taking into account the fact that $y'(t)\leq 0$, we can multiply (4.16) by $(-y'(t))$ and integrate over (t,T^*). We get:

$$y(t)^{1+\eta} \leq C\cdot(-y'(t))^{2+\eta-p},\ \forall\ t\in[0,T^*),$$

and thanks to the definition of T^* it follows that:

$$y'(t)+\frac{1}{C}y(t)^{\frac{1+\eta}{2+\eta-p}} \leq 0, \quad \forall\ t\in[0,+\infty).$$

Hence (4.4) holds for $p\neq 1$.
(ii) The case $p=1$. Instead of (1.7) we can use this time (1.6). Thus, for all $s\in(t,T)$, we have:

$$\|\nabla u(s)\|_{\infty}^{-\eta} \geq C_4(s-t)^{\eta/2}(M(t)-m(t))^{-\eta}. \tag{4.17}$$

Combining relations (4.7), (4.17) and (4.9) we get:

$$\int_t^T(s-t)^{\eta/2}(M(s)-m(s))^{\gamma}\,ds \leq C_5(M(t)-m(t))^{\gamma} \text{ for all } t\in[0,T] \tag{4.18}$$

where C_5 is a positive constant which depends only on N,a,p,β,η and Ω.
Thanks to (4.3) we have $\eta>2$. As previously, we fix $\delta\in(0,T^*)$, then, using (1.4), (1.5), (4.3) and (4.18) we get:

$$\begin{aligned}&\int_0^{\infty}(1+s)(M(s)-m(s))^{\gamma}\,ds =\\ &\quad=\int_0^{\delta}(1+s)(M(s)-m(s))^{\gamma}\,ds+\int_{\delta}^{T^*}(1+s)(M(s)-m(s))^{\gamma}\,ds\\ &\quad\leq(1+\delta)[\delta(M(0)-m(0))^{\gamma}+\delta^{-\eta/2}\int_0^{T^*}s^{\eta/2}(M(s)-m(s))^{\gamma}\,ds].\end{aligned}$$

¿From (1.4), (1.5) and (4.18), this last integral is finite. Consequently relation (4.12) is valid for $p = 1$, too. As in the first case we deduce that the function y is well defined on $[0, +\infty)$ and belongs to $W^{2,\infty}((0, +\infty))$, the first and the second derivatives being given by (4.13) and (4.14). Since $\eta > 2$, using Hölder inequality we get this time:

$$\begin{aligned}(y(t))^{\eta/2} &= (\int_t^\infty (s-t)(M(s)-m(s))^\gamma\, ds)^{\eta/2} \\ &\le \Big[\int_t^{T^*} (s-t)^{\eta/2}(M(s)-m(s))^\gamma\, ds\Big]\Big[\int_t^{T^*} (M(s)-m(s))^\gamma\, ds\Big]^{\eta/2-1}\end{aligned}$$

Taking into account (4.18), (4.13) and (4.14) we deduce:

$$y(t)^{\eta/2} \le C \cdot y''(t)(-y'(t))^{\eta/2-1} \tag{4.19}$$

and by the same arguments as previously we get:

$$y'(t) + \frac{1}{C} y(t) \le 0 \text{ for all } t \in [0, +\infty)$$

which ends the proof of Proposition 4.1, as $p = 1$. □

Proof of Theorem 1.3: Let:

$$y(t) = \int_t^\infty (s-t)(M(s)-m(s))^\gamma\, ds, \quad t \in [0, \infty),$$

be the function defined in Proposition 4.1. We have obtained that $y \in W^{2,+\infty}((0, +\infty))$ and there is a positive constant C depending only on p, β, N, Ω and $(M(0) - m(0))$ such that y satisfies the differential inequality (4.4):

$$y'(t) + Cy(t)^\alpha \le 0, \text{ for all } t \in [0, +\infty),$$

with α given by (4.5) and $y(0) = \int_0^\infty s(M(s)-m(s))^\gamma\, ds \ge 0$.

On the one hand if $p \in (0, 1)$ then $\alpha \in (0, 1)$ and thanks to (4.4) and (4.6) we get that $T^* < \infty$ and:

$$y(t) \equiv 0 \text{ for } t > T^*$$

Consequently, for $p \in (0, 1)$, the extinction of the gradient in finite time of the solution to problem (1.1) occurs.

On the other hand, if $p \ge 1$ then $\alpha \ge 1$ and thanks to (4.4) and from the fact that y is a positive function, we deduce that:

$$\frac{y'(t)}{y^\alpha(t)} \le -\frac{1}{C} \quad \text{for all} \quad t \in (0, +\infty). \tag{4.20}$$

We distinguish between the two cases below:
(i) The case $p > 1$. We have $\alpha > 1$ and integrating (4.20), over $(0,t)$, $t > 0$, we obtain:

$$y(t)^{1-\alpha} \geq y(0)^{1-\alpha} + \frac{(\alpha-1)t}{C},$$

or else:

$$y(t) \leq \left(\frac{1}{y(0)^{1-\alpha} + \frac{(\alpha-1)t}{C}} \right)^{1/(\alpha-1)}.$$

(ii) The case $p = 1$. We have $\alpha = 1$ and integrating (4.20) over $(0,t)$, $t > 0$ we get this time:

$$\log y(t) \leq \log y(0) - \frac{t}{C},$$

or else:

$$y(t) \leq y(0)e^{-\frac{t}{C}}.$$

Thus, we have obtained the decreasing rate for the function y, as $t \to \infty$.
We claim that:

$$\lim_{t\to\infty} M(t) - m(t) = 0, \tag{4.21}$$

which implies that, for $p \geq 1$, the solution u of problem (1.1) converges uniformly in $\overline{\Omega}$ to a constant, as $t \to \infty$.
To prove (4.21) we recall that, the function defined by:

$$g(t) = (1+t)(M(t) - m(t))^{\gamma} \text{ belongs to } L^1(0, +\infty).$$

Since $t \mapsto (M(t) - m(t))$ is a positive and decreasing function on $[0, +\infty)$, there exists a positive constant c such that:

$$c = \lim_{t\to\infty} (M(t) - m(t))^{\gamma}.$$

Then, $c(1+t) \leq g(t)$, and it follows that the function $t \to c(1+t)$ belongs to $L^1(0, +\infty)$, which is possible only if $c = 0$. So assertion (4.21) holds. Now, we want to find a decreasing rate for the application $t \mapsto (M(t) - m(t))$. Denote by f the decreasing rate of the function y:

$$f(t) = \begin{cases} y(0)e^{-\frac{t}{C}} \text{ if } p = 1, \\ \left(\frac{1}{y(0)^{(1-\alpha)} + \frac{(\alpha-1)t}{C}} \right)^{1/(\alpha-1)} \quad \text{if } p > 1. \end{cases} \tag{4.22}$$

Then, for all $t > 0$ we have:

$$\int_{t/2}^{t} (s - t/2)(M(s) - m(s))^{\gamma}\, ds \leq y(t/2) \leq f(t/2).$$

Since $s \mapsto (M(s) - m(s))$ is a decreasing function, we deduce that

$$(M(t) - m(t))^\gamma \int_{t/2}^{t} (s - t/2)\, ds \leq f(t/2), \quad \forall\, t > 0,$$

which implies:

$$M(t) - m(t) \leq \left(\frac{8f(t/2)}{t^2} \right)^{\frac{1}{\gamma}}, \quad \forall\, t > 0, \tag{4.23}$$

where f is given by (4.22). This ends the proof of Theorem 1.3. $\square$

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